

# The second cohomology of the homological Goldman Lie algebra

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## Abstract

We determine the second cohomology group of the homological Goldman Lie algebra for an oriented surface.

## 1 Introduction

By a *surface*, we mean an oriented compact connected two-dimensional smooth manifold with boundary. The first homology group of a surface and its intersection form reflect the topological structure of the surface. For example, they have information about the genus and the boundary components of the surface.

To study them in details, we consider a Lie algebra coming from them. We call it the homological Goldman Lie algebra of the first homology group of the surface. Goldman introduced the Lie algebra for study of the moduli space of  $GL_1(\mathbb{R})$ -flat bundles over the surface [1]p.295-p.297. More precisely, the homological Goldman Lie algebra is the subalgebra of all Fourier polynomials in the Poisson algebra on the moduli space if the surface is closed. Moreover, Goldman introduced more complicated Lie algebra coming from free loops and the local intersection number, which is called the Goldman Lie algebra. We have a surjective homomorphism from the Goldman Lie algebra onto the homological Goldman Lie algebra. Thus, the Lie algebras are interesting in a geometrical context.

The homological Goldman Lie algebra is interesting in not only a geometrical context but also an algebraic context because it is infinite dimensional

and we can define this algebra only from algebraic information. The propose of this paper is to study the algebraic structure of the homological Goldman Lie algebra. In preceding paper [3], we determined the ideals of the homological Goldman Lie algebra. Moreover, the algebra is a graded Lie algebra and the grading induces a grading of an ideal of the homological Goldman Lie algebra. We determine the second cohomology groups of the homological Goldman Lie algebra and its derived Lie subalgebra. The cohomology of a Lie algebra  $\mathfrak{g}$  over a commutative ring  $R$  is the graded cohomology group of the cochain complex  $C^*(\mathfrak{g}) = C^*(\mathfrak{g}; R)$ , where we denote by  $C^p(\mathfrak{g})$  the  $R$ -vector space of alternating  $p$ -multilinear forms on  $\mathfrak{g}$ , for example see [2]p.592.

Let  $H$  be a  $\mathbb{Z}$ -module, i.e., an abelian group, and  $\langle -, - \rangle : H \times H \rightarrow \mathbb{Z}$ ,  $(x, y) \mapsto \langle x, y \rangle$ , an alternating  $\mathbb{Z}$ -bilinear form. For example, we consider  $H$  the first homology group, and  $\langle -, - \rangle$  the intersection form on  $H$  of a surface. We define a  $\mathbb{Z}$ -linear map  $\mu : H \rightarrow \text{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$  by  $\mu(x)(y) = \langle x, y \rangle$ . Denote by  $\mathbb{Q}H$  the  $\mathbb{Q}$ -vector space with basis the set  $H$ ;

$$\mathbb{Q}H := \left\{ \sum_{i=1}^n c_i [x_i] \mid n \in \mathbb{N}, c_i \in \mathbb{Q}, x_i \in H \right\},$$

where  $[-] : H \rightarrow \mathbb{Q}H$  is the embedding as basis. We remark  $c[x] \neq [cx] \in \mathbb{Q}H$  if  $c \neq 1$ . For  $x, y \in H$ , we define a bracket  $[-, -] : \mathbb{Q}H \times \mathbb{Q}H \rightarrow \mathbb{Q}H$  by  $[[x], [y]] := \langle x, y \rangle [x + y]$ . It is easy to see that this bracket is skew and satisfies the Jacobi identity [1]p.295-p.297. The Lie algebra  $(\mathbb{Q}H, [-, -])$  is called the *homological Goldman Lie algebra of  $(H, \langle -, - \rangle)$* . We have obtained  $H^1(\mathbb{Q}H) \cong \text{Hom}_{\mathbb{Q}}(\text{Ker } \mu, \mathbb{Q})$  by [3]. Our main theorem in the present paper is

**Theorem 1.1.** *We have a natural  $\mathbb{Q}$ -linear isomorphism*

$$\text{Hom}_{\mathbb{Z}}(H, \mathbb{Q}) \rightarrow H^2(\mathbb{Q}H)$$

*if  $\langle -, - \rangle$  is non-degenerate.*

For example, the intersection form  $\langle -, - \rangle$  on the first homology group is non-degenerate if the surface is closed. Hence, we have the isomorphism  $\text{Hom}_{\mathbb{Z}}(H_1(\Sigma; \mathbb{Z}), \mathbb{Q}) \cong H^2(\mathbb{Q}H_1(\Sigma; \mathbb{Z}))$  if the surface  $\Sigma$  is closed. In this paper, we obtain the result also in the case  $\langle -, - \rangle$  is degenerate.

## 2 Derived algebra of some subalgebra

Let  $S$  be a subset of  $H$ . Then we define  $S^{(1)} := \{u+v \in H \mid u, v \in S, \langle u, v \rangle \neq 0\}$ . For example, we have

$$H^{(1)} = H \setminus \ker \mu.$$

In fact, assume  $x \in H^{(1)}$ . Then there exist  $u$  and  $v \in H$  with  $\langle u, v \rangle \neq 0$  and  $x = u + v$ . Since  $\langle x, v \rangle = \langle u, v \rangle \neq 0$ , we have  $x \in H \setminus \ker \mu$ . Conversely, assume  $x \in H \setminus \ker \mu$ . Then there exists  $y \in H$  with  $\langle x, y \rangle \neq 0$ . Since  $\langle x - y, y \rangle = \langle x, y \rangle \neq 0$  and  $x = (x - y) + y$ , we have  $x \in H^{(1)}$ .

**Propositon 2.1.**  $\mathbb{Q}S$  is a subalgebra of  $\mathbb{Q}H$  if and only if  $S^{(1)} \subset S$ . Then,  $(\mathbb{Q}S)^{(1)} = [\mathbb{Q}S, \mathbb{Q}S] = \mathbb{Q}(S^{(1)})$ .

*Proof.* Assume  $\mathbb{Q}S$  is a subalgebra of  $\mathbb{Q}H$ . For  $x \in S^{(1)}$ , there exist  $u$  and  $v \in S$  with  $x = u + v$  and  $\langle u, v \rangle \neq 0$ . We have  $[x] = \frac{1}{\langle u, v \rangle} [[u], [v]] \in [\mathbb{Q}S, \mathbb{Q}S] \subset \mathbb{Q}S$ . Hence, we have  $x \in S$ . This proves  $S^{(1)} \subset S$  and  $\mathbb{Q}(S^{(1)}) \subset [\mathbb{Q}S, \mathbb{Q}S]$ .

Assume  $S^{(1)} \subset S$ . It is trivial  $\mathbb{Q}S$  is a subspace of  $\mathbb{Q}H$  as  $\mathbb{Q}$ -vector space. For  $u$  and  $v \in S$ , we have

$$[[u], [v]] = \begin{cases} 0, & \text{if } \langle u, v \rangle = 0, \\ \langle u, v \rangle [u + v], & \text{if } \langle u, v \rangle \neq 0. \end{cases}$$

In both cases we have  $[[u], [v]] \in \mathbb{Q}(S^{(1)}) \subset \mathbb{Q}S$ . Hence,  $\mathbb{Q}S$  is a subalgebra of  $\mathbb{Q}H$ . And this shows  $[\mathbb{Q}S, \mathbb{Q}S] \subset \mathbb{Q}(S^{(1)})$ .  $\square$

The inclusion  $\mathbb{Q}H^{(1)} \rightarrow \mathbb{Q}H$  induces the restriction  $\rho^{(1),p} : C^p(\mathbb{Q}H) \rightarrow C^p(\mathbb{Q}H^{(1)})$ . On the other hand, we can decompose  $\mathbb{Q}H$  into the center and the derived subalgebra,  $\mathbb{Q}H = \mathbb{Q}\ker \mu \oplus \mathbb{Q}H^{(1)}$ . See [3]. The projection  $\mathbb{Q}H \rightarrow \mathbb{Q}H^{(1)}$  of the decomposition induces the zero extension  $\varepsilon^{(1),p} : C^p(\mathbb{Q}H^{(1)}) \rightarrow C^p(\mathbb{Q}H)$ . The maps  $\rho^{(1)} = (\rho^{(1),p})_{p=1}^\infty$  and  $\varepsilon^{(1)} = (\varepsilon^{(1),p})_{p=1}^\infty$  are cochain maps and  $\rho^{(1)} \circ \varepsilon^{(1)} = id$ . Then, the restriction  $H^p(\rho^{(1)}) : H^p(\mathbb{Q}S) \rightarrow H^p(\mathbb{Q}S^{(1)})$  is a surjective  $\mathbb{Q}$ -linear map.

## 3 Decomposition of the cohomology

Let  $S$  be a subset of  $H$  with  $S^{(1)} \subset S$ . For  $p > 0$  and  $z \in H$ , we define the subspace  $C^p(\mathbb{Q}S)_{(z)}$  of  $C^p(\mathbb{Q}S)$  by

$$C^p(\mathbb{Q}S)_{(z)} := \left\{ \omega \in C^p(\mathbb{Q}S) \mid \begin{array}{l} u_1, \dots, u_p \in S, u_1 + \dots + u_p \neq z \\ \implies \omega([u_1], \dots, [u_p]) = 0 \end{array} \right\}.$$

The subspace  $C^*(\mathbb{Q}S)_{(z)}$  is a subcomplex of the cochain complex  $C^*(\mathbb{Q}S)$ , that is, we have  $d(C^p(\mathbb{Q}S)_{(z)}) \subset C^{p+1}(\mathbb{Q}S)_{(z)}$  for all  $p > 0$ . In fact, for  $\omega \in C^p(\mathbb{Q}S)_{(z)}$  and  $u_1, \dots, u_{p+1} \in S$  with  $u_1 + \dots + u_{p+1} \neq z$ ,

$$\begin{aligned} & d\omega([u_1] \cdots, [u_{p+1}]) \\ &= \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega(\langle u_i, u_j \rangle [u_i + u_j], [u_1], \dots, [\hat{u}_i], \dots, [\hat{u}_j], \dots, [u_{p+1}]) \\ &= 0 \end{aligned}$$

since  $\omega \in C^p(\mathbb{Q}S)_{(z)}$  and  $(u_i + u_j) + u_1 + \dots + \hat{u}_i + \dots + \hat{u}_j + \dots + u_{p+1} \neq z$ .

Define the restriction  $\rho_{(z)}^p : C^p(\mathbb{Q}S) \rightarrow C^p(\mathbb{Q}S)_{(z)}$  by

$$\rho_{(z)}^p(\omega)([u_1], \dots, [u_p]) = \begin{cases} \omega([u_1], \dots, [u_p]), & u_1 + \dots + u_p = z, \\ 0, & u_1 + \dots + u_p \neq z. \end{cases}$$

We can define the decomposition map  $\rho^p : C^p(\mathbb{Q}S) \rightarrow \prod_{z \in H} C^p(\mathbb{Q}S)_{(z)}$ , by  $\rho^p = \prod_{z \in H} \rho_{(z)}^p$ . We define the gluing map  $\gamma^p : \prod_{z \in H} C^p(\mathbb{Q}S)_{(z)} \rightarrow C^p(\mathbb{Q}H)$  by

$$\gamma^p((\omega_z)_{z \in H})([u_1], \dots, [u_p]) := \omega_{u_1 + \dots + u_p}([u_1], \dots, [u_p]).$$

The maps  $\rho_{(z)} = (\rho_{(z)}^p)_{p=1}^\infty$ ,  $\gamma = (\gamma^p)_{p=1}^\infty$  and  $\rho = (\rho^p)_{p=1}^\infty$  are well-defined and cochain maps. Moreover,  $\gamma$  is the inverse of  $\rho$ . Hence, we have  $Z^p(\mathbb{Q}S) \cong \prod_{z \in H} Z^p(\mathbb{Q}S)_{(z)}$ ,  $B^p(\mathbb{Q}S) \cong \prod_{z \in H} B^p(\mathbb{Q}S)_{(z)}$  and  $H^p(\mathbb{Q}S) \cong \prod_{z \in H} H^p(\mathbb{Q}S)_{(z)}$ , where we define  $Z^p(\mathbb{Q}S)_{(z)} := Z^p(C^*(\mathbb{Q}S)_{(z)})$ ,  $B^p(\mathbb{Q}S)_{(z)} := B^p(C^*(\mathbb{Q}S)_{(z)})$  and  $H^p(\mathbb{Q}S)_{(z)} := H^p(C^*(\mathbb{Q}S)_{(z)})$ .

We call  $H^p(\mathbb{Q}S)_{(z)}$  an *inner component* if  $z \in \ker \mu$ , and an *outer component* if  $z \in H^{(1)}$ .

## 4 Inner component

In this section, we fix an element  $z \in \ker \mu$ .

We define the subspace  $\hat{C}^p(\mathbb{Q}H)_{(z)}$  of  $C^p(\mathbb{Q}H)_{(z)}$  by the following condition;  $\omega \in \hat{C}^p(\mathbb{Q}H)_{(z)}$  if and only if the map  $H^{p-1} = H \times \dots \times H \ni (u_1, \dots, u_{p-1}) \mapsto \omega([u_1], \dots, [u_{p-1}], [z - u_1 - \dots - u_{p-1}]) \in \mathbb{Q}$  is a multilinear map.

**Propositon 4.1.** *We have  $d(\hat{C}^p(\mathbb{Q}H)_{(z)}) = 0$ . In particular, the subspace  $\hat{C}^p(\mathbb{Q}H)_{(z)}$  is a subcomplex of the cochain complex  $C^*(\mathbb{Q}H)_{(z)}$ , and we have  $H^p(\hat{C}^*(\mathbb{Q}H)_{(z)}) = Z^p(\hat{C}^*(\mathbb{Q}H)_{(z)}) = \hat{C}^p(\mathbb{Q}H)_{(z)}$ .*

*Proof.* For  $\omega \in \hat{C}^p(\mathbb{Q}H)_{(z)}$  and  $u_1, \dots, u_{p+1} \in H$  with  $u_1 + \dots + u_{p+1} = z$ ,

$$\begin{aligned}
& d\omega([u_1], \dots, [u_p], [u_{p+1}]) \\
&= \sum_{1 \leq i < j \leq p} (-1)^{i+j} \langle u_i, u_j \rangle \omega([u_i + u_j], [u_1], \dots, [\hat{u}_i], \dots, [\hat{u}_j], \dots, [u_p], [u_{p+1}]) \\
&\quad + \sum_{k=1}^p (-1)^{k+p+1} \langle u_k, u_{p+1} \rangle \omega([u_k + u_{p+1}], [u_1], \dots, [\hat{u}_k], \dots, [u_p]) \\
&= \sum_{1 \leq i < j \leq p} (-1)^{i+j} \langle u_i, u_j \rangle \omega([u_i], [u_1], \dots, [\hat{u}_i], \dots, [\hat{u}_j], \dots, [u_p], [u_j + u_{p+1}]) \\
&\quad + \sum_{1 \leq i < j \leq p} (-1)^{i+j} \langle u_i, u_j \rangle \omega([u_j], [u_1], \dots, [\hat{u}_i], \dots, [\hat{u}_j], \dots, [u_p], [u_i + u_{p+1}]) \\
&\quad + \sum_{k=1}^p (-1)^{k+1} \langle u_k, z - \sum_{\ell=1}^p u_\ell \rangle \omega([u_1], \dots, [\hat{u}_k], \dots, [u_p], [u_k + u_{p+1}]) \\
&= \sum_{1 \leq i < j \leq p} (-1)^{j-1} \langle u_i, u_j \rangle \omega([u_1], \dots, [\hat{u}_j], \dots, [u_p], [u_j + u_{p+1}]) \\
&\quad + \sum_{1 \leq i < j \leq p} (-1)^i \langle u_i, u_j \rangle \omega([u_1], \dots, [\hat{u}_i], \dots, [u_p], [u_i + u_{p+1}]) \\
&\quad + \sum_{k=1}^p \sum_{\ell=1}^p (-1)^{k+1} \langle u_k, u_\ell \rangle \omega([u_1], \dots, [\hat{u}_k], \dots, [u_p], [u_k + u_{p+1}]) \\
&= \sum_{1 \leq i < j \leq p} (-1)^{j-1} \langle u_i, u_j \rangle \omega([u_1], \dots, [\hat{u}_j], \dots, [u_p], [u_j + u_{p+1}]) \\
&\quad + \sum_{1 \leq i < j \leq p} (-1)^i \langle u_i, u_j \rangle \omega([u_1], \dots, [\hat{u}_i], \dots, [u_p], [u_i + u_{p+1}]) \\
&\quad + \sum_{1 \leq i < j \leq p} (-1)^{i+1} \langle u_i, u_j \rangle \omega([u_1], \dots, [\hat{u}_i], \dots, [u_i + u_{p+1}]) \\
&\quad + \sum_{1 \leq i < j \leq p} (-1)^{j+1} \langle u_j, u_i \rangle \omega([u_1], \dots, [\hat{u}_j], \dots, [u_j + u_{p+1}]) \\
&= 0.
\end{aligned}$$

□

**Propositon 4.2.** *We have a natural isomorphism*

$$\text{Hom}_{\mathbb{Z}}(\wedge^{p-1} H / \mathbb{Z}z, \mathbb{Q}) \rightarrow \hat{C}^p(\mathbb{Q}H)_{(z)}$$

*as  $\mathbb{Q}$ -vector spaces.*

*Proof.* The isomorphism is given by the equation

$$f(u_1, \dots, u_{p-1}) = \omega([u_1], \dots, [u_{p-1}], [z - u_1 - \dots - u_{p-1}]) \quad (1)$$

for  $f \in \text{Hom}_{\mathbb{Z}}(\wedge^{p-1} H / \mathbb{Z}z, \mathbb{Q})$  and  $\omega \in \hat{C}^p(\mathbb{Q}H)_{(z)}$ . For  $f \in \text{Hom}_{\mathbb{Z}}(\wedge^{p-1} H / \mathbb{Z}z, \mathbb{Q})$ , we define  $\omega \in \hat{C}^p(\mathbb{Q}H)_{(z)}$  by the equation (1). This is well-defined. In fact, for  $u_1, \dots, u_p \in H$  with  $u_1 + \dots + u_p = z$ ,

$$\begin{aligned} \omega([u_p], [u_2], \dots, [u_{p-1}], [u_1]) &= f(u_p, u_2, \dots, u_{p-1}) \\ &= f(z, u_2, \dots, u_{p-1}) - \sum_{i=1}^{p-1} f(u_i, u_2, \dots, u_{p-1}) \\ &= -f(u_1, \dots, u_{p-1}) \\ &= -\omega([u_1], \dots, [u_{p-1}], [u_p]). \end{aligned}$$

Conversely, for  $\omega \in \hat{C}^p(\mathbb{Q}H)_{(z)}$ , we define  $f \in \text{Hom}_{\mathbb{Z}}(\wedge^{p-1} H / \mathbb{Z}z, \mathbb{Q})$  by the equation (1). This is also well-defined. In fact, for  $u_2, \dots, u_{p-1} \in H$ ,

$$\begin{aligned} f(z, u_2, \dots, u_{p-1}) &= \omega([z], [u_2], \dots, [u_{p-1}], [-u_2 - \dots - u_{p-1}]) \\ &= -\omega([-u_2 - \dots - u_{p-1}], [u_2], \dots, [u_{p-1}], [z]) \\ &= \sum_{i=2}^{p-1} \omega([u_i], [u_2], \dots, [u_{p-1}], [z - u_i]) \\ &= 0. \end{aligned}$$

This proves the proposition.  $\square$

Therefore, we have the inclusion

$$\varphi^p : \text{Hom}_{\mathbb{Z}}(\wedge^{p-1} H / \mathbb{Z}z, \mathbb{Q}) = H^p(\hat{C}^*(\mathbb{Q}H)_{(z)}) \rightarrow H^p(\mathbb{Q}H)_{(z)}.$$

We consider the composition  $H^p(\rho_{(z)} \circ \rho^{(1)}) \circ \varphi^p$ . The map  $H^p(\rho_{(z)} \circ \rho^{(1)}) \circ \varphi^p$  is injective if and only if  $\hat{C}^p(\mathbb{Q}H)_{(z)} \cap (\rho_{(z)} \circ \rho^{(1),p})^{-1}(B^p(\mathbb{Q}H^{(1)})_{(z)}) = 0$ .

**Theorem 4.3.** *The maps  $H^2(\rho_{(z)} \circ \rho^{(1)}) \circ \varphi^2$  and  $\varphi^2$  are injective if  $H^{(1)} \neq \emptyset$ .*

*Proof.* Choose and fix  $x_0 \in H^{(1)}$ . Take  $\omega \in \hat{C}^2(\mathbb{Q}H)_{(z)} \cap (\rho^{(1),2})^{-1}(B^2(\mathbb{Q}H^{(1)})_{(z)})$ . Then, there exists  $\eta \in C^1(\mathbb{Q}H^{(1)})_{(z)}$  with  $(\rho_{(z)} \circ \rho^{(1),2})(\omega) = d\eta$ . For  $x \in H^{(1)}$ , we have

$$\omega([x], [z-x]) = \rho_{(z)} \circ \rho^{(1),2}(\omega)([x], [z-x]) = d\eta([x], [z-x]) = -\langle x, z-x \rangle \eta([z]) = 0.$$

For  $x \in \ker \mu$ , we have

$$\omega([x], [z - x]) = \omega([x + x_0], [z - x - x_0]) - \omega([x_0], [z - x_0]) = 0.$$

Hence, we obtain  $\omega = 0$ .

The injectivity of  $\varphi^2$  follows immediately from that of  $H^2(\rho_{(z)} \circ \rho^{(1)}) \circ \varphi^2$ .  $\square$

**Theorem 4.4.** *The map*

$$H^2(\rho_{(z)} \circ \rho^{(1)}) \circ \varphi^2 : \text{Hom}_{\mathbb{Z}}(\wedge^{p-1} H / \mathbb{Z}z, \mathbb{Q}) \rightarrow H^2(\mathbb{Q}H^{(1)})_{(z)}$$

*is surjective.*

To prove this theorem, we prepare some lemmas.

**Lemma 4.5** (Key lemma). *If  $x_1, \dots, x_n \in H^{(1)} = H \setminus \ker \mu$ , there exists  $z \in H$  satisfying the condition  $\langle x_1, z \rangle \neq 0, \dots$  and  $\langle x_n, z \rangle \neq 0$ .*

*Proof.* We prove this by induction on  $n$ .

It is clear in the case  $n = 1$ . Consider the case  $n > 1$ . Take  $u \in H$  satisfying the condition  $\langle x_i, u \rangle \neq 0$  for  $i = 1, \dots, n - 1$ . If  $\langle x_n, u \rangle \neq 0$ , there is nothing to do. Suppose not. We can choose  $v \in H$  such that  $\langle x_n, v \rangle \neq 0$ , since  $x_n \notin \ker \mu$ . We shall prove that

$$z := u + (1 + |\langle x_1, u \rangle| + \dots + |\langle x_{n-1}, u \rangle|)v$$

is a desired one. We have

$$\langle x_n, z \rangle = (1 + |\langle x_1, u \rangle| + \dots + |\langle x_{n-1}, u \rangle|) \langle x_n, v \rangle \neq 0.$$

For  $k < n$ ,  $\langle x_k, z \rangle = \langle x_k, u \rangle \neq 0$  if  $\langle x_k, v \rangle = 0$ .

If not,  $\langle x_k, z \rangle \neq 0$ , because

$$|\langle x_k, z \rangle| \geq (1 + |\langle x_1, u \rangle| + \dots + |\langle x_{n-1}, u \rangle|) |\langle x_k, v \rangle| - |\langle x_k, u \rangle| > 0.$$

$\square$

Take  $\omega \in Z^2(\mathbb{Q}H) \cup Z^2(\mathbb{Q}H^{(1)})$ , and define the function  $f$  by  $f(x) = \omega([x], [z - x])$ . We study the property of the function.

**Lemma 4.6.** *For all  $x$  and  $y \in H^{(1)}$  with  $x + y \in H^{(1)}$ , we have  $f(x + y) = f(x) + f(y)$ .*

*Proof.* In the case  $\langle x, y \rangle \neq 0$ , the lemma holds since

$$\begin{aligned} 0 &= d\omega([x], [y], [z - x - y]) \\ &= \langle x, y \rangle (-\omega([x + y], [z - x - y]) + \omega([y], [z - y]) + \omega([x], [z - x])) \\ &= \langle x, y \rangle (-f(x + y) + f(x) + f(y)). \end{aligned}$$

Suppose  $\langle x, y \rangle = 0$ . By Lemma 4.5, we can choose  $u \in H^{(1)}$  with  $\langle x, u \rangle \neq 0$ ,  $\langle y, u \rangle \neq 0$  and  $\langle x + y, u \rangle \neq 0$ .

$$f(x + y) = f(x + y + u) - f(u) = f(x) + f(y + u) - f(u) = f(x) + f(y)$$

since  $\langle x + y, u \rangle \neq 0$ ,  $\langle x, y + u \rangle = \langle x, u \rangle \neq 0$  and  $\langle y, u \rangle \neq 0$ . This proves the lemma.  $\square$

**Lemma 4.7.** *For  $x \in H^{(1)}$  and  $c \in \mathbb{Z} \setminus 0$ , we have  $f(cx) = cf(x)$ .*

*Proof.* By Lemma 4.6, it is sufficient to show when  $c = -1$ . By Lemma 4.5, we can choose  $y \in H$  with  $\langle x, y \rangle \neq 0$ . We have  $f(y) = f(-x) + f(y + x)$  and  $f(y + x) = f(x) + f(y)$  by Lemma 4.6. Hence, we have  $f(-x) = f(y) - f(y + x) = -f(x)$ .  $\square$

Define the map  $F : H^{(1)} \times \ker \mu \rightarrow \mathbb{Q}$  by  $F(x, v) = f(x + v) - f(x)$ .

**Lemma 4.8.** *The map  $F$  satisfies the following conditions.*

1.  $F$  is independent of the variable  $x \in H^{(1)}$ .
2.  $F$  is a  $\mathbb{Z}$ -linear map in the variable  $v \in \ker \mu$ .
3.  $F(x, z) = 0$

*Proof.* For  $x$  and  $y \in H^{(1)}$  and  $v \in \ker \mu$ ,

$$\begin{aligned} F(x, v) - F(y, v) &= f(x + v) - f(x) - f(y + v) + f(y) \\ &= \begin{cases} f(x + y + v) - f(x + y + v), & \text{if } x + y \in H^{(1)}, \\ f(x - y) - f(x - y), & \text{if } x - y \in H^{(1)}, \end{cases} \\ &= 0. \end{aligned}$$

Hence, the map  $F$  is independent of  $x \in H^{(1)}$ .



For  $x \in H^{(1)}$  and  $v$  and  $w \in \ker \mu$ ,

$$\begin{aligned}
F(x, v+w) &= F(2x, v+w) \\
&= f(2x+v+w) - f(2x) \\
&= f(x+v) + f(x+w) - 2f(x) \\
&= (f(x+v) - f(x)) + (f(x+w) - f(x)) \\
&= F(x, v) + F(x, w).
\end{aligned}$$

Hence, the map  $F$  is  $\mathbb{Z}$ -linear map in  $v \in \ker \mu$ .

For  $x \in H^{(1)}$ ,

$$\begin{aligned}
F(x, z) &= F(x-z, z) = f(x) - f(x-z) = f(x) + f(z-x) \\
&= \omega([x], [z-x]) + \omega([z-x], [x]) = 0.
\end{aligned}$$

□

*Proof of Theorem 4.4.* If  $H^{(1)} = \emptyset$ , we have  $H^2(\mathbb{Q}H^{(1)})_{(z)} \subset H^2(\mathbb{Q}H^{(1)}) = 0$  since  $\mathbb{Q}H^{(1)} = 0$ . Hence the theorem is trivial.

Next, we assume  $H^{(1)} \neq \emptyset$  and take  $x_0 \in H^{(1)}$ . We can define the inverse  $\psi : H^2(\mathbb{Q}H^{(1)})_{(z)} \rightarrow \text{Hom}_{\mathbb{Z}}(H/\mathbb{Z}z, \mathbb{Q})$  by

$$\psi([\omega])(x) := \begin{cases} f(x), & \text{if } x \in H^{(1)}, \\ F(x_0, x), & \text{if } x \in \ker \mu, \end{cases}$$

where  $f(x) = \omega([x], [z-x])$  and  $F(x_0, x) = f(x_0+x) - f(x_0)$ . This is well-defined. In fact, we have  $B^2(\mathbb{Q}H^{(1)})_{(z)} = 0$  since

$$d\eta([x], [z-x]) = -\omega(\langle x, z-x \rangle [z]) = 0.$$

We prove  $\psi([\omega]) \in \text{Hom}_{\mathbb{Z}}(H/\mathbb{Z}z, \mathbb{Q})$ , or equivalently the linearity of  $\psi([\omega])$ . For  $x$  and  $y \in \ker \mu$ , we have

$$\psi([\omega])(x+y) = F(x_0, x+y) = F(x_0, x) + F(x_0, y) = \psi([\omega])(x) + \psi([\omega])(y).$$

For  $x$  and  $y \in H^{(1)}$  with  $x+y \in H^{(1)}$ , we have

$$\psi([\omega])(x+y) = f(x+y) = f(x) + f(y) = \psi([\omega])(x) + \psi([\omega])(y).$$

For  $x$  and  $y \in H^{(1)}$  with  $x+y \in \ker \mu$ , we have

$$\begin{aligned}
\psi([\omega])(x+y) &= F(x_0, x+y) = F(-y, x+y) = f(x) - f(-y) \\
&= f(x) + f(y) = \psi([\omega])(x) + \psi([\omega])(y).
\end{aligned}$$

For  $x \in H^{(1)}$  and  $y \in \ker \mu$ ,

$$\begin{aligned}\psi([\omega])(x+y) &= f(x+y) = f(x) + F(x,y) \\ &= f(x) + F(x_0, y) = \psi([\omega])(x) + \psi([\omega])(y).\end{aligned}$$

Moreover we have

$$\psi([\omega])(z) = F(x_0, z) = 0.$$

Thus we obtain  $\psi([\omega]) \in \text{Hom}_{\mathbb{Z}}(H/\mathbb{Z}z, \mathbb{Q})$ . Finally, we check  $(H^2(\rho_{(z)} \circ \rho^{(1)}) \circ \varphi^2) \circ \psi = \text{id}$ . For  $\omega \in Z^2(\mathbb{Q}H^{(1)})$ ,

$$H^2(\rho_{(z)} \circ \rho^{(1)}) \circ \varphi^2 \circ \psi([\omega]) = \varphi^2 \circ \psi([\omega]) = \varphi^2(f) = [\omega].$$

This completes the proof of Theorem 4.4.  $\square$

We remark this  $\psi$  is the inverse of  $H^2(\rho_{(z)} \circ \rho^{(1)}) \circ \varphi^2$  when  $H^{(1)} \neq \emptyset$ . By Theorem 4.3 and Theorem 4.4, we have the isomorphism

$$H^2(\mathbb{Q}H^{(1)})_{(z)} \cong \text{Hom}_{\mathbb{Z}}(H/\mathbb{Z}z, \mathbb{Q})$$

if  $H^{(1)} \neq \emptyset$ . Moreover, since the above lemmas hold when  $\omega \in Z^2(\mathbb{Q}H)$  and Theorem 4.3 holds for  $\varphi^2$ , we have the isomorphism

$$H^2(\mathbb{Q}H)_{(z)} \cong H^2(\mathbb{Q}\ker \mu)_{(z)} \oplus H^2(\mathbb{Q}H^{(1)})_{(z)}$$

if  $H^{(1)} \neq \emptyset$ .

## 5 Outer component

**Propositon 5.1.** *If  $H^{p-1}(\mathbb{Q}H)_{(z)} = 0$  for  $z \in H^{(1)}$ , then the restriction  $H^p(\rho^{(1)}) : H^p(\mathbb{Q}H)_{(z)} \rightarrow H^p(\mathbb{Q}H^{(1)})_{(z)}$  is injective for  $z \in H^{(1)}$ .*

*Proof.* Assume  $\omega \in Z^p(\mathbb{Q}H)_{(z)}$  satisfies  $\omega|_{\mathbb{Q}H^{(1)}} \in B^p(\mathbb{Q}H^{(1)})_{(z)}$ . Then there exists  $\eta \in C^{p-1}(\mathbb{Q}H^{(1)})$  with  $\omega|_{\mathbb{Q}H^{(1)}} = d\eta$ . We denote by  $L$  the Lie derivative and by  $i$  the inner product. For  $v \in \ker \mu$ , we have

$$0 = L([v])\omega = (d \circ i([v]) + i([v]) \circ d)(\omega) = d(i([v])(\omega))$$

since  $[v] \in \mathfrak{z}(\mathbb{Q}H)$ , the center of  $\mathbb{Q}H$ . Hence, there exists  $\eta_v \in C^{p-2}(\mathbb{Q}H)_{(z-v)}$  with  $i([v])(\omega) = d\eta_v$  since  $H^{p-1}(\mathbb{Q}H)_{(z-v)} = 0$ . Define  $\zeta_v \in C^1(\mathbb{Q}H)_{(v)}$  by  $\zeta_v([v]) = 1$ . We get  $\zeta_v \wedge \eta_v \in C^{p-1}(\mathbb{Q}H)_{(z)}$ . Remark  $d\zeta_v = 0$  since  $[v] \in$

$\mathfrak{z}(\mathbb{Q}H)$ . The sum  $\sum_{v \in \ker \mu} \zeta_v \wedge \eta_v$  is well-defined. In fact,  $\zeta_v \wedge \eta_v([u_1], \dots, [u_{p-1}]) = 0$  for  $u_1, \dots, u_{p-1} \in H$  if  $v \neq u_i$  for  $i = 1, \dots, p-1$ . Thus,  $\sum_{v \in \ker \mu} \zeta_v \wedge \eta_v$  is a finite sum for any  $u_1, \dots, u_{p-1} \in H$ . We have

$$d\left(\sum_{v \in \ker \mu} \zeta_v \wedge \eta_v\right) = \sum_{v \in \ker \mu} (d\zeta \wedge \eta_v - \zeta_v \wedge d\eta_v) = - \sum_{v \in \ker \mu} \zeta_v \wedge i([v])(\omega).$$

Here we have

$$\begin{aligned} & (\zeta_v \wedge i([v])(\omega))([u_1], \dots, [u_p]) \\ &= \begin{cases} (-1)^i \omega([v], [u_1], \dots, [\hat{u}_i], \dots, [u_p]) = \omega([u_1], \dots, [u_p]), & \text{if there exists unique } i \text{ with } u_i = v, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence

$$d\left(\sum_{v \in \ker \mu} \zeta_v \wedge \eta_v\right)([u_1], \dots, [u_p]) = -\#\{u_i \mid u_i \in \ker \mu\} \omega([u_1], \dots, [u_p]).$$

Define  $\theta \in C^{p-1}(\mathbb{Q}H)_{(z)}$  by

$$\begin{aligned} & \theta([u_1], \dots, [u_{p-1}]) \\ &= \begin{cases} \eta([u_1], \dots, [u_{p-1}]), & \text{if } u_i \in H^{(1)} \text{ for } i = 1, \dots, p-1, \\ \frac{-1}{\#\{u_i \mid u_i \in \ker \mu\}} (\sum_{v \in \ker \mu} \zeta_v \wedge \eta_v)([u_1], \dots, [u_{p-1}]), & \text{otherwise.} \end{cases} \end{aligned}$$

It is easy to check  $\theta \in C^{p-1}(\mathbb{Q}H)_{(z)}$  and  $d\theta = \omega$ . Hence, the proposition holds.  $\square$

**Theorem 5.2.** For  $p = 1, 2$  and  $z \in H^{(1)}$ , we have  $H^p(\mathbb{Q}H)_{(z)} = 0$  and  $H^p(\mathbb{Q}H^{(1)})_{(z)} = 0$ .

*Proof.* It suffices to show  $H^p(\mathbb{Q}H)_{(z)} = 0$  since the restriction  $H^p(\rho^{(1)})_{(z)} : H^p(\mathbb{Q}H)_{(z)} \rightarrow H^p(\mathbb{Q}H^{(1)})_{(z)}$  is surjective. Take and fix  $y \in H^{(1)}$  with  $\langle y, z \rangle \neq 0$ . Let  $C^0(\mathbb{Q}H)_{(z)} = 0$ . Define the maps  $\Phi^p : C^p(\mathbb{Q}H)_{(z)} \rightarrow C^{p-1}(\mathbb{Q}H)_{(z)}$  by  $\Phi^1 = 0$ ,  $\Phi^2(\omega)([z]) = -\frac{1}{\langle y, z \rangle} \omega([y], [z - y])$  and

$$\begin{aligned} \Phi^3(\omega)([u_1], [u_2]) &= -\frac{1}{\langle y, z \rangle} (\omega([y], [u_1 - y], [u_2]) + \omega([y], [u_1], [u_2 - y])) \\ &\quad + \frac{1}{2\langle y, z \rangle} \omega([2y], [u_1 - y], [u_2 - y]) \\ &\quad + \frac{\langle u_1 - y, u_2 - y \rangle}{2\langle y, z \rangle^2} \omega([y], [2y], [u_1 + u_2 - 3y]). \end{aligned}$$

We can check  $d^{p-1} \circ \Phi^p + \Phi^{p+1} \circ d^p = id_{C^p(\mathbb{Q}H)}$ . Hence, the theorem holds.  $\square$

**Remark 5.3.** We have already constructed  $\Phi^4$  and  $\Phi^5$  in a similar ways. As a corollary, we have  $H^p(\mathbb{Q}H)_{(z)} = 0$  and  $H^p(\mathbb{Q}H^{(1)})_{(z)} = 0$  for  $p = 3$  and  $p = 4$ . We will discuss the detail elsewhere.

Combining the Theorem 5.2 with the results the section 4, we have the natural isomorphism

$$H^2(\mathbb{Q}H) \cong \prod_{z \in \ker \mu} \text{Hom}_{\mathbb{Z}}(H/\mathbb{Z}z, \mathbb{Q}) \cong \text{Hom}_{\mathbb{Z}} \left( \prod_{z \in \ker \mu} H/\mathbb{Z}z, \mathbb{Q} \right)$$

if  $H^{(1)} \neq \emptyset$ . In particular, when  $\langle -, - \rangle$  is non-degenerate, that is,  $\ker \mu = 0$ , we have the natural isomorphism

$$\varphi : H^2(\mathbb{Q}H) \cong \text{Hom}_{\mathbb{Z}}(H, \mathbb{Q}). \quad (2)$$

This holds also when  $H = 0$ .

## 6 The subalgebra $\ker(\mathbb{Q}H \rightarrow \mathbb{Q} \otimes H)$

Define the  $\mathbb{Q}$ -linear map  $\alpha : \mathbb{Q}H \rightarrow \mathbb{Q} \otimes H$  by  $\alpha(\sum_{i=1}^n c_i[x_i]) = \sum_{i=1}^n c_i \otimes x_i$ . Let  $\mathfrak{g}$  be  $\ker \alpha$ .

**Propositon 6.1.** *The subspace  $\mathfrak{g}$  is a subalgebra of  $\mathbb{Q}H$ .*

*Proof.* Take  $X$  and  $Y \in \mathfrak{g}$ . We represent  $X = \sum_{i=1}^m c_i[x_i]$  and  $Y = \sum_{j=1}^n d_j[y_j]$  with  $\sum_{i=1}^m c_i \otimes x_i = 0$  and  $\sum_{j=1}^n d_j \otimes y_j = 0$ . We extend the alternating  $\mathbb{Z}$ -bilinear form  $\langle -, - \rangle : H \times H \rightarrow \mathbb{Z}$  to the alternating  $\mathbb{Q}$ -bilinear form  $\langle -, - \rangle_{\mathbb{Q}} : (\mathbb{Q} \otimes H) \times (\mathbb{Q} \otimes H) \rightarrow \mathbb{Q} \otimes \mathbb{Z} \rightarrow \mathbb{Q}$  with  $\langle x, y \rangle = 1 \langle 1 \otimes x, 1 \otimes y \rangle_{\mathbb{Q}}$  for  $x$  and  $y \in H$ . We have  $[X, Y] \in \mathfrak{g}$  because

$$\begin{aligned} \alpha([X, Y]) &= \alpha\left(\sum_{i=1}^m \sum_{j=1}^n c_i d_j \langle x_i, y_j \rangle [x_i + y_j]\right) \\ &= \sum_{i=1}^m \sum_{j=1}^n c_i d_j \langle x_i, y_j \rangle \otimes (x_i + y_j) \\ &= \sum_{i=1}^m \langle c_i \otimes x_i, \sum_{j=1}^n d_j \otimes y_j \rangle_{\mathbb{Q}} \otimes x_i + \sum_{j=1}^n \langle \sum_{i=1}^m c_i \otimes x_i, d_j \otimes y_j \rangle_{\mathbb{Q}} \otimes y_j \\ &= 0. \end{aligned}$$

□

**Propositon 6.2.** *The restriction  $H^2(\mathbb{Q}H) \rightarrow H^2(\mathfrak{g})$  is injective if  $\langle -, - \rangle$  is non-degenerate.*

*Proof.* We have the isomorphism  $\varphi : \text{Hom}_{\mathbb{Z}}(H, \mathbb{Q}) \rightarrow H^2(\mathbb{Q}H)$  in (2) since  $\langle -, - \rangle$  is non-degenerate. Take  $\omega \in Z^2(\mathbb{Q}H)$  with  $[\omega|_{\mathfrak{g}}] = 0$  and let  $(\varphi)^{-1}([\omega]) = f$ . It is enough to show  $f(x) = 0$  for  $x \in H \setminus 0$ .

We may assume  $\omega([u], [v]) = 0$  if  $u + v \neq 0$  because  $H^2(\mathbb{Q}H)_{(z)} = 0$  for  $z \in H^{(1)} = H \setminus 0$ . By  $[\omega|_{\mathfrak{g}}] = 0$ , there exists  $\eta \in C^1(\mathfrak{g})$  with  $\omega|_{\mathfrak{g}} = d\eta$ . Then we have

$$\begin{aligned} 0 &= -\eta(0) = -\eta([2x] - 2[x], [-2x] - 2[-x]) \\ &= d\eta([2x] - 2[x], [-2x] - 2[-x]) \\ &= \omega|_{\mathfrak{g}}([2x] - 2[x], [-2x] - 2[-x]) \\ &= \omega([2x], [-2x]) - 2\omega([x], [-2x]) - 2\omega([2x], [-x]) + 4\omega([x], [-x]) \\ &= f(2x) + 4f(x) = 6f(x). \end{aligned}$$

Hence the proposition holds. □

## References

- [1] W. M. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface groups representations, *Invent. Math.* **85**, 263-302 (1986).
- [2] G. P. Hochschild and J. P. Serre, Cohomology of Lie algebras, *Ann. of Math.* **57** (1953), 591-603.
- [3] K. Toda, The ideals of the homological Goldman Lie algebra, preprint, arXiv: 1112.1213 (2011)

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